International Journal of Engineering, Science and Mathematics
Vol. 6 Issue 8, December 2017 (Special Issue)
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijesm.co.in, Email: ijesmj@gmail.com Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

Numerical Solution of Typical Non-linear Parabolic Partial Differential Equations using Haar Wavelets

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Abstract

In this paper, we proposed an efficient Haar wavelet collocation method for the numerical solution of typical nonlinear parabolic partial differential equations. Such type of problem arises in various fields of science and engineering. In the present study more accurate solutions have been obtained by Haar wavelet decomposition with multiresolution analysis. Test problem is considered to check the efficiency and accuracy of the proposed method. An extensive amount of error analysis has been carried out to obtain the convergence of the method. The numerical results are found in good agreement with exact and finite difference method (FDM), which shows that the solution using Haar wavelet collocation method (HWCM) is more effective and accurate and manageable for this type of problems.

Keywords:

Haar wavelet, collocation method; non-linear parabolic PDE, difference method, Numerical simulation; Error analysis.;

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1. Introduction

Most scientific problems arise in real-world physical problems such as plasma physics, fluid mechanics, solid state physics and in many branches of chemistry [13]. We know that except a limited number of these problems, most of them do not have analytical solutions. The importance of obtaining the approximate solutions of nonlinear partial differential equations in physics and mathematics is still a significant problem that needs new methods to discover approximate solutions. Therefore, these nonlinear equations should be solved using numerical methods i.e. variational iteration method (VIM) [8] and homotopy-perturbation method (HPM) [7]. The BBM equation was introduced by Benjamin-Bona-Mahony, as an improvement of the Ko-rtewegde Vries equation (KdV equation) in 1+1 dimensions. The Benjamin-Bona-Mahony (BBM) equation is inherently of

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nonlinearity, which models long waves in a nonlinear dispersive system, and as a result, the BBM equation incorporates dispersive effects. Recently, various authors have been proposed different methods for the solution of different type of BBM equations [9, 14].
In numerical analysis, Wavelets are used as appropriate tools at various places to provide good mathematical model for scientific phenomena, which are usually modeled through linear or nonlinear differential equations. Haar wavelet method is one of them because of Haar functions appearing very attractive in many applications. The previous work in wavelet analysis via Haar wavelets was led by Chen and Hsiao [4], who first derived a Haar operational matrix for the integrals of the Haar function and put the application for the Haar analysis into the dynamic systems. In order to take the advantages of the local property, many authors researched the Haar wavelet to solve the differential and integral equations [1-3, 10,11].
The objective of the present work is to apply the Haar wavelet collocation method (HWCM) for the numerical solution of different types of Benjamin-Bona-Mahony equations and obtained results are compared with the classical FDM and exact solution. The present method is illustrated by some of the Benjamin-Bona-Mahony equations.
The present paper is organized as follows; in section 2, Haar wavelets and its generalized operational matrix of integration are given. Haar Wavelet Collocation Method for solving Benjamin-Bona-Mahony equations is presented in section 3. Section 4 deals with the numerical Experiment, results and error analysis of the illustrative problems. Finally, conclusion of the proposed work is discussed in section 5.

1. Haar wavelets and Operational matrix of integration

We used the simplest wavelet function i.e Haar wavelet. We establish an operational matrix for integration via Haar wavelets. The scaling function $h_1(x)$ for the family of the Haar wavelets is defined as

$$h_1(x) = \begin{cases} 1, & for\ x \in [0,1) \\ 0, & Otherwise \end{cases} \tag{2.1}$$

The Haar wavelet family for $x \in [0,1)$ is defined as

$$h_i(x) = \begin{cases} 1, & for\ x \in \left[\dfrac{k}{m}, \dfrac{k+0.5}{m}\right) \\ -1, & for\ x \in \left[\dfrac{k+0.5}{m}, \dfrac{k+1}{m}\right) \\ 0, & Otherwise \end{cases} \tag{2.2}$$

In the above definition the integer $m = 2^l$, $l = 1, 2, ..., J$, indicates the level of resolution and integer $k = 0, 1, 2, ..., m-1$ is the translation parameter. Maximum level of resolution is J. The index i in Eq. (2.2) is calculated using $i = m + k + 1$. In case of minimal values $m = 1$, $k = 0$, then $i = 2$. The maximal value of i is $N = 2^{J+1}$.
Let us define the grid points $x_j = (j - 0.5)/N$, $j = 1, 2, ..., N$, discretize the Haar function $h_i(x)$, in this way we get Haar coefficient matrix $H(i, j) = h_i(x_j)$ which has the dimension $N \times N$. The operational matrix of integration is obtained by integrating (2.2) is as

$$Ph_i(x) = \int_0^x h_i(x)\,dx \tag{2.3}$$

$$Qh_i(x) = \int_0^x Ph_i(x)\,dx \tag{2.4}$$

and

$$Ch_i(x) = \int_0^1 Ph_i(x)\,dx \tag{2.5}$$

These integrals can be evaluated by using equation (2.2) and they are given by

$$Ph_i(x)=\begin{cases} x-\dfrac{k}{m}, & for\ \ x\in\left[\dfrac{k}{m},\dfrac{k+0.5}{m}\right) \\ \dfrac{k+1}{m}-x, & for\ \ x\in\left[\dfrac{k+0.5}{m},\dfrac{k+1}{m}\right) \\ 0, & Otherwise \end{cases} \tag{2.6}$$

$$Qh_i(x)=\begin{cases} \dfrac{1}{2}\left(x-\dfrac{k}{m}\right)^2, & for\ \ x\in\left[\dfrac{k}{m},\dfrac{k+0.5}{m}\right) \\ \dfrac{1}{4m^2}-\dfrac{1}{2}\left(\dfrac{k+1}{m}-x\right)^2, & for\ \ x\in\left[\dfrac{k+0.5}{m},\dfrac{k+1}{m}\right) \\ \dfrac{1}{4m^2}, & for\ \ x\in\left[\dfrac{k+1}{m},1\right) \\ 0, & Otherwise \end{cases} \tag{2.7}$$

and

$$Ch_i(x)=\begin{cases} \dfrac{1}{2}\left(1-\dfrac{k}{m}\right)^2, & for\ \ x\in\left[\dfrac{k}{m},\dfrac{k+0.5}{m}\right) \\ \dfrac{1}{4m^2}-\dfrac{1}{2}\left(\dfrac{k+1}{m}-1\right)^2, & for\ \ x\in\left[\dfrac{k+0.5}{m},\dfrac{k+1}{m}\right) \\ \dfrac{1}{4m^2}, & for\ \ x\in\left[\dfrac{k+1}{m},1\right) \\ 0, & Otherwise \end{cases} \tag{2.8}$$

For instance, $J=2$ $\Rightarrow$ N = 8, then from (2.2), (2.6), (2.7) & (2.8) we have

$$H(8,8)=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

$$Ph(8,8)=\frac{1}{16}\begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 \\ 1 & 3 & 5 & 7 & 7 & 5 & 3 & 1 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$Qh(8,8)=\frac{1}{512}\begin{pmatrix} 1 & 9 & 25 & 49 & 81 & 121 & 169 & 225 \\ 1 & 9 & 25 & 49 & 79 & 103 & 119 & 127 \\ 1 & 9 & 23 & 31 & 32 & 32 & 32 & 32 \\ 0 & 0 & 0 & 0 & 1 & 9 & 23 & 31 \\ 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

and $$Ch(8,8)=\frac{1}{64}\begin{pmatrix} 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 \\ 32 & 32 & 32 & 32 & 16 & 16 & 16 & 16 \\ 32 & 32 & -4 & -4 & 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 & 8 & 8 & 4 & 4 \\ 32 & -17 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 18 & -7 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 8 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$

3. Haar Wavelet Collocation Method for solving Benjamin-Bona-Mahony equations

Consider the general Benjamin-Bona-Mahony equation of the form:

$$u_t+\alpha u_x+\beta uu_x-u_{xxt}=\phi(x,t) \tag{3.1}$$

with the initial conditions $u(x,0)=f(x),\ 0\le x<1$ (3.2)

and the boundary condition $u(0,t)=g_0(t),\ u(1,t)=g_1(t) \quad t>0$ (3.3)

where $\phi(x,t),\ f(x),\ g_0(t),\ g_1(t)$ are functions of independent variables and α,β are constants.

Let us assume that $$\dot{u}''(x,t)=\sum_{i=1}^{N}a_ih_i(x) \tag{3.4}$$

where $.\ and\ '$ are differentiation w. r. t. t and x respectively & a_i's, $i=1,2,...,N$ are Haar coefficients to be determined.

Now integrating the equation (3.4) once w.r.t. t from t_s to t

$$u''(x,t)-u''(x,t_s)=(t-t_s)\sum_{i=1}^{N}a_ih_i(x)$$

$$\Rightarrow u''(x,t)=(t-t_s)\sum_{i=1}^{N}a_ih_i(x)+u''(x,t_s) \tag{3.5}$$

where t_s is the initial time and $\Delta t=t-t_s$ is the time interval.

Also integrating (3.5) w. r. t. x from 0 to x, we get

$$u'(x,t)-u'(0,t)=\Delta t\sum_{i=1}^{N}a_iPh_i(x)+u'(x,t_s)-u'(0,t_s)$$

$$u'(x,t)=\Delta t\sum_{i=1}^{N}a_iPh_i(x)+u'(x,t_s)+u'(0,t)-u'(0,t_s) \tag{3.6}$$

and again integrating equation (3.6) w. r. t. x

$$u(x,t)-u(0,t)=\Delta t\sum_{i=1}^{N}a_iQh_i(x)+u(x,t_s)-u(0,t_s)+x(u'(0,t)-u'(0,t_s)) \tag{3.7}$$

$$u(x,t)=\Delta t\sum_{i=1}^{N}a_iQh_i(x)+u(x,t_s)+u(0,t)-u(0,t_s)+x(u'(0,t)-u'(0,t_s)) \tag{3.8}$$

Put $x=1$ in (3.8) and by using equation (3.3) (i.e. boundary conditions), we get

$$u(1,t)=\Delta t\sum_{i=1}^{N}a_iCh_i(x)+u(1,t_s)+u(0,t)-u(0,t_s)+(u'(0,t)-u'(0,t_s))$$

$$\Rightarrow u'(0,t)-u'(0,t_s)=g_1(t)-\Delta t\sum_{i=1}^{N}a_iCh_i(x)-g_1(t_s)-g_0(t)+g_0(t_s) \tag{3.9}$$

Substituting the equation (3.9) in (3.8), then the equation (3.8) becomes

$$u\,(x,t)=\Delta t\sum_{i=1}^{N}a_iQh_i(x)+u(x,t_s)+g_0\,(t)-g_0\,(t_s)+$$
$$x\left\{g_1(t)-\Delta t\sum_{i=1}^{N}a_iCh_i(x)-g_1(t_s)-g_0(t)+g_0(t_s)\right\} \quad (3.10)$$

Differentiating equation (3.10) w. r. t. t then we have

$$\dot{u}(x,t)=\sum_{i=1}^{N}a_iQh_i(x)+\dot{g}_0\,(t)+x\left\{\dot{g}_1\,(t)-\sum_{i=1}^{N}a_iCh_i(x)-\dot{g}_0\,(t)\right\} \quad (3.11)$$

Substituting equations (3.4), (3.6), (3.10) in (3.11) in equation (3.1) and by solving using Inexact Newton's method [12], we obtain the Haar wavelet coefficients $a_i\,'s$. Substituting these values of $a_i\,'s$ in (3.10), we get the HWCM based numerical solution of the given problem (3.1). The error will be calculated by $L_\infty=\max\left|u(x,t)_e-u(x,t)_a\right|$, where $u(x,t)_e$ and $u(x,t)_a$ are exact and approximate solutions respectively.

4. Numerical Experiment

In this section, we apply the HWCM discussed in section 3 to some of the parabolic partial differential equations.

Problem 1. Consider the linear BBM equation of the form [6]

$$u_t-2u_{xxt}+u_x=0 \quad (4.1)$$

with initial condition: $u(x,0)=e^{-x}\;,\;0\le x<1$ (4.2)

and boundary conditions: $u(0,\,t)=e^{-t},\quad u(1,\,t)=e^{-1-t}\quad t>0$ (4.3)

Let $\dot{u}''=\sum_{i=1}^{N}a_ih_i(x)$ (4.4)

Using (4.2) & (4.3), then the equations (3.4), (3.5), (3.10) and (3.11) becomes

$$\dot{u}''(x,t)=\sum_{i=1}^{N}a_ih_i(x) \quad (4.5)$$

$$u'(x,t)=\Delta t\sum_{i=1}^{N}a_iPh_i(x)+e^{-x}-e^{-t}+1 \quad (4.6)$$

$$u\,(x,t)=\Delta t\sum_{i=1}^{N}a_iQh_i(x)+e^{-x}+e^{-t}-1+$$
$$x\left\{e^{-1-t}-\Delta t\sum_{i=1}^{N}a_iCh_i(x)-e^{-1}-e^{-t}+1\right\} \quad (4.7)$$

$$\dot{u}(x,t)=\sum_{i=1}^{N}a_iQh_i(x)-e^{-t}+x\left\{-e^{-1-t}-\sum_{i=1}^{N}a_iCh_i(x)+e^{-t}\right\} \quad (4.8)$$

Substituting (4.5), (4.6) and (4.8) in (4.1), we get

$$\sum_{i=1}^{N}a_iQh_i(x)-e^{-t}+x\left\{-e^{-1-t}-\sum_{i=1}^{N}a_iCh_i(x)+e^{-t}\right\}-$$
$$2\sum_{i=1}^{N}a_ih_i(x)+\left[\Delta t\sum_{i=1}^{N}a_iPh_i(x)+e^{-x}-e^{-t}+1\right]=0 \quad (4.9)$$

By solving (4.9), we get the Haar wavelet coefficients $a_i\,'s$ using Inexact Newton's method [5]. i.e. [-0.55, -0.14, -0.09, -0.05, -0.05, -0.04, -0.03, -0.02]. Substituting these $a_i\,'s$ in (4.7), we obtain the HWCM based numerical solution of the equation (4.1) and is compared with the FDM and exact solution $u(x,t)=e^{-x-t}$ in

Table 1 for $N= 8$ and Fig. 1 for N=32. The error analysis for higher values of N is given in Table 2 with $\Delta t = 1/N$.

Problem 2. Next consider the linear non-homogenous BBM equation of the form [12]

$$u_t - 2u_{xxt} = -e^{x+t} \tag{4.10}$$

with initial condition: $u(x,0) = e^x \, , \, 0 \le x < 1$ (4.11) and

boundary conditions: $u(0,t) = e^t, \quad u(1,t) = e^{1+t} \, , \quad t > 0$ (4.12)

Using (4.11) & (4.12), then the equations (3.4), (3.10) and (3.11) becomes

$$\dot{u}''(x,t) = \sum_{i=1}^{N} a_i h_i(x)$$

$$u\,(x,t) = \Delta t \sum_{i=1}^{N} a_i Q h_i(x) + e^x + e^t - 1 + x\left\{ e^{1+t} - \Delta t \sum_{i=1}^{N} a_i C h_i(x) - e^1 - e^t + 1 \right\}$$

(4.13)

(4.14)

$$\dot{u}(x,t) = \sum_{i=1}^{N} a_i Q h_i(x) + e^t + x\left\{ e^{1+t} - \sum_{i=1}^{N} a_i C h_i(x) - e^t \right\}$$

(4.15)

Substituting (4.13and (4.15) in (4.10), we get

$$\sum_{i=1}^{N} a_i Q h_i(x) + e^t + x\left\{ e^{1+t} - \sum_{i=1}^{N} a_i C h_i(x) - e^t \right\} - 2\sum_{i=1}^{N} a_i h_i(x) + e^{x+t} = 0$$

(4.16)

Equation (4.16) can be solved, we get the Haar wavelet coefficients a_i's using Inexact Newton's method [12]. We obtain Haar coefficients a_i's i.e. [1.95, -0.47, -0.18, -0.30,-0.08, -0.10, -0.13, -0.17]. Substituting these a_i's in (4.14), we get the HWCM based numerical solution of the equation (4.10) and is compared with the FDM & exact solution $u(x,t) = e^{x+t}$ in Table 3 for $N= 8$ and Fig. 2 for $N=32$. The error analysis for higher values of N is given in Table 4 with $\Delta t = 1/N$.

Problem 3. Now consider the non-linear BBM equation [6]

$$u_t - u_{xxt} + u u_x = 0 \tag{4.17}$$

with initial condition: $u(x,0) = x \, , \, 0 \le x < 1$ (4.18)

and boundary conditions: $u(0,t) = 0, \quad u(1,t) = \dfrac{1}{1+t} \, , \quad t > 0$ (4.19)

Using the conditions (4.18) & (4.19), the equation (4.17) can be solved as explained in Section (3) and we get Haar coefficients a_i's i.e. [0.05, -0.03, -0.01, -0.02, -0.01, -0.01, -0.01, -0.01]. Substituting these values of a_i's in (3.10), we obtain the HWCM based numerical solution of the equation (4.17) and is compared with the FDM & exact solution $u(x,t) = x/(1+t)$ in Table 5 for N= 8 and Fig. 3 for N=32. The error analysis for higher values of N is given in Table 6 with $\Delta t = 1/N$.

5. Conclusion

In the present study, numerical solutions of Benjamin-Bona-Mahoney (BBM) equations are discussed using Haar wavelet collocation method. The proposed method is computationally efficient and the algorithm can be easily implemented on computer, which has been justified through the illustrative problems. The numerical solutions are presented in Tables and figures, from which we observed that the Haar solutions are very good in agreement with exact solutions and finite difference method. The comparison with analytical solution shows that Haar wavelet gives better results with less computational cost: it is due to the sparsity of the transform matrix and small number of wavelet coefficients and HWCM is better than classical numerical methods viz. FDM. Subsequently error analysis is presented, which shows that the accuracy of the solution is increased by increasing the number of grid points (i.e. N). It is worth mentioning that Haar wavelet provides excellent results for small and large values of N. Hence the proposed method is very effective and easy to implement for solving linear as well as non-linear Benjamin-Bona-Mahoney (BBM) equations.

Acknowledgment

It is a pleasure to thank the University Grants Commission (UGC), Govt. of India for the financial support under UGC-SAP DRS-III for 2016-2021:F.510/3/DRS-III/2016(SAP-I) Dated: 29th Feb. 2016.

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